Positional Tolerance
And
Relative Accuracy

Text: Surveying Measurements and their Analysis
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February 20, 2014

A land surveyor is, by act of the state legislature, qualified to make measurements of the land or lands below water. Those measurements, together with the knowledge base of interpreting maps and deeds, lead to the establishment of laying out deed lines between property owners. By licensure, he is qualified to provide expert testimony regarding measurements in courts of law. Therefore, he must be in control of those measurements.

Errors occur in every measurement made by a person or made by an instrument manufactured by a person. Common standards of practice afford the surveyor basic skills in measurement. By analyzing measurements and acknowledging the presence of error, the surveyor will become more confident in his ability to exercise his professional judgment and will be able to qualify every measurement made to a reasonable degree of accuracy and precision.

Some Basic Definitions:

- Land Surveying – The science and art of making such measurements as are necessary to determine the relative position of points above, on, or beneath the surface of the earth, or to establish such points in a specified position.
- Measurement – The act or process of ascertaining the extent, dimensions, or quantity of something. An estimate of a quantity. Anything measured is inexact.
- Count – To check over one by one to determine the total number. An exact quantity of a sample or population.
- Blunder – A careless mistake.
- Error – A deviation from accuracy or correctness. The difference between an observed or computed value and the true value.
- Plane Surveying – Surveying under the assumption that the earth is a plane surface and that all north–south lines are parallel.
- Geodetic Surveying – Surveying considering the earth's curvature and convergence of meridians.

Units of Measurement:

Linear Units – Predetermined lengths in one dimension to determine distances.
(Inches, Feet, Yard, Rod, Chain, Mile, Meter.)
Angular Units – Predetermined portions of a circle to determine angles.
(Seconds, Minutes, Degrees, Radians, Grads.)
Area Units – Predetermined distances in two dimensions to determine area.
(Acre, Hectare.)

Types of Errors:

- Systematic Errors – Errors conforming to known mathematical and physical laws. They are predictable and remain the same under set conditions but may vary in magnitude.
- Random Errors – An error whose presence is unavoidable and unpredictable but generally behaves according to mathematical laws and tends to cancel but never completely do.

Precision and Accuracy:

- Precision – The agreement among readings of the same quantity. Precision relates to the refinement in manufacture of equipment and the care and refinement in making measurements. If precision in measurement is high, the random errors should be small. Precision relates to the method of measurement.
- Accuracy – The agreement of the measurement or measurements with the true value. A value that is closer to the true value is more accurate than one which is farther than the true. Accuracy relates to the result.

Sources of Errors in Land Surveying:

- Natural Errors – Caused by effects from nature, including temperature, humidity, gravity, atmospheric pressure, atmospheric refraction, curvature of the earth, wind, tension, etc.
- Instrumental Errors – Caused by either the initial manufacture of a measuring instrument or by wear and//or maladjustment of the measuring instrument. Initial manufacturing errors are predictable and are, therefore, systematic. Wear and/or maladjustment errors are unpredictable and are, therefore, random.
- Personal Errors – Caused by the inability of humans to perceive anything exact, including readings, aligning cross-hairs and other marks or centering devices. Aside from this innate inability which affects all of us, people vary in their manual dexterity, experience, training, intelligence, motivation and the desire to employ appropriate care.

Direct and Indirect Measurements:

- Direct Measurements – Measuring directly between points or lines.
- Indirect Measurements – Measurement computed from other measurements.

Significant Figures:

- Or Significant Digits – The number of digits that are meaningful in a measurement or quantity.
  a. Zeroes used merely to indicate the position of the decimal point are not significant. For example, the number 0.00618 has three significant figures.
  b. If zeroes are recorded at the end of a measurement, they are significant. For example, the number 61.410 has five significant figures. The number 0.0060 has two significant figures.
  c. Zeroes between non-zero digits are significant. For example, the number 12.1003 has six significant figures.
  d. In a number ending with one or more zeroes to the left of the decimal, a special indication of the exact number of significant figures must be made.

For example, the number 615,000 has six significant figures. The number 615,000 has three significant figures. The number 360 could have either two or three significant figures.

e. Truncating a number means to delete all digits to the right of the number of significant figures. For example, the number 3.141592654 truncated to five significant figures is 3.1415.

f. Rounding a number means to either change the last number of significant figure up one if the number is greater than 5 or down one if the number is less than or equal to 5. For example, the number 3.141592654 rounded to five significant figures is 3.1416.

g. The precision of a number, lacking other methods of determining precision, is generally considered to be plus or minus ½ of the last place. For example, the number 42.81 is understood to be 42.81 plus or minus 0.005 or an uncertainty of plus or minus 0.005.

h. If a number has an uncertainty value, one can discover how many significant figures are contained in a measured quantity. If the number 42.81 has an uncertainty value of plus or minus 0.5, then the number should end in the units' column or it should be stated as 43.

- Adding or Subtracting – The number of significant figures in the sum or difference is determined by the fewest decimal places in the numbers added or subtracted. For example 14.623 + 12.01 + 1.0 = 27.633 but since one of the numbers in the sum has only one place to the right of the decimal, the sum can only be significant to one place to the right of the decimal or 27.6.
- Multiplying or Dividing – The number of significant figures in the product or quotient is determined by the fewest number of significant figures in the values used. For example, 14.29 x 0.051 = 0.73, because 0.051 has two significant figures and 14.29 has four significant figures, the product may only have two.
- Conversion factors or constants are not measured quantities, and thus do not determine significant figures. For example, 1342.5 inches / 12 inches/foot + 111.87 feet (five significant figures in the number of inches yields five significant figures in the quotient.
- To avoid round-off errors when using conversion factors or constants that contain a large or infinite number of digits (or are not counts) use one extra figure in such values. For instance, a trigonometric number or the constant Pi. For example, using Pi, multiply 165.41 by Pi. It would be incorrect to multiply 165.41 x 3.14 = 519.39. It would be correct to multiply 165.41 x 3.14159 = 519.65 since the product has no round-off error.
- To avoid round-off errors when using computed values in subsequent calculations, carry one extra figure throughout the intermediate calculations. For example, 131.46 x 32.68 = 4296 as a final answer, but use 4296.1 in any further calculations.
- Round off final answers to the significant figures warranted by the measurements and the rules on computing as cited above.

Errors in Angular Measurements:

a. Reading Error – The inability to read a vernier or circle to some precision.
b. Pointing Error – The inability to place the cross-hairs in the telescope exactly centered on a target.
c. Instrument Centering Error – The inability to place a theodolite or transit directly over a point.
d. Target Centering Error – The inability to place a target directly over a point.
e. Instrument Leveling Error – The inability to perfectly level a theodolite or transit.

Errors in Distance Measurements:

- Generally are associated with the ability to read a measuring tape or a manufacturer's standard error.

Propagation of Random Errors:

- Error in a Sum – Where each error is different, for example, in an angular analysis. The total error is the square root of the sum of the squares.
- Error in a Series – Where each error is the same, for example, in a taped distance of multiple lengths of tape. The total error is the error multiplied by the square root of the number of times the error occurs.
- Error in a Product – Similar to an error in a sum, for example, given uncertainty in the measurements on the sides of a rectangle, the error in a product will yield the uncertainty associated in the computed area.

Statistics:

- Statistics is that branch of the mathematical sciences that analyses a random sample and provides conclusions for an entire population.
- Statistical theory relies on several assumptions:
  1. The random sample that is analyzed is random.
  2. The random sample is sufficiently large to provide accurate data for theoretical conclusions.
  3. The random sample is a true representation of the entire population.
  4. Any errors in the random sample are random.

Definitions of Statistical Terms:

- Sample Size – The number of observations or measurements in a sample.
- Mean – Or arithmetic mean, the sum of the observations or measurements of a sample divided by the sample size.
- Median – The middle value of a sample.
- Mode – The value which occurs most frequently in a sample.
- Residual – The arithmetic difference between an individual value in a sample and the mean of that sample.
- Probability Curve – A plot of points resembling the shape of a bell centered on the mean value of a sample. Generally, the Y-axis of the plot is where the mean value is plotted.
- Points of Inflexion – The points on the probability curve where the curve changes shape from concave down to concave up. Given that the area under the probability

curve and above the X-axis is 100% of the total area, the area under the probability curve between two lines drawn perpendicular to the X-axis through the points of inflexion equals 68.3% of the total area.

- Standard Deviation – Or Standard Deviation of a Single Value, the number representing the distance on the X-axis from the mean to the points of inflexion. Theoretically, the standard deviation is symmetric about the mean value. It is equal to the square root of the sum of the squared residuals divided by the sample size minus one. Generally, it is represented by the Greek letter, sigma.
- Standard Error of the Mean – An uncertainty statement regarding the average (mean value) of a set rather than a randomly selected single value, as the standard deviation is. The uncertainty is with respect to the true value, not the mean value. It is equal to the standard deviation divided by the square root of the sample size. The value could be positive or negative.
- Level of Certainty – Or Percent Probability, it represents a level or degree of confidence regarding an error or uncertainty statement. One multiplied by the standard deviation, or one-sigma, is a 68.3% level of certainty. Two multiplied by the standard deviation, or two-sigma, is a 95% level of certainty. Three multiplied by the standard deviation, or three-sigma, is a 99.7% level of certainty.

2005 ALTA/ACSM Accuracy Standards

Definitions:

Relative Positional Accuracy – The value expressed in feet or meters that represents the uncertainty due to random errors in measurements in the location of any point on a survey relative to any other point on the same survey at the 95 percent confidence level.

Surveying Measurements and their Analysis

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Positional Tolerance or positional uncertainty must always:

- Be stated with a percent certainty (such as 90%, 95%, etc.)
- Be relative to something fixed in position.

"Statements are incomplete if a reference such as a control point or coordinate system origin is not given. For example, it is incomplete to state that *'positions are correct ±0.02 feet'* or *'points are accurately located ±0.xx feet'* It is theoretically impossible to achieve such implied absolute accuracy in three dimensional space having no origin and with an implication of 100% certainty. Therefore, such a statement is meaningless, creates much confusion, and is impossible to check." Chapter 8 gave procedures to derive positional uncertainty of interrelated stations. Such theories must be used in standards of practice to lend meaning to them.

RENTON VOCATIONAL TECHNICAL INSTITUTE
MATHEMATICS FOR LAND SURVEYORS

Mark Harrison March 1990

STATISTICS FOR LAND SURVEYORS

Statistics is that branch of the Mathematical Sciences that analyses a random sample and provides conclusions for an entire population. It is used by many professional disciplines, including the Geodetic Surveyors, and is being introduced to Boundary/Plane Surveyors of Washington State. It will be understood by professional land surveyors throughout the State by the mid 1990's.

Statistical theory relies on several assumptions:

a) The random sample that is analyzed is random.

b) The random sample is sufficiently large to provide accurate data for theoretical conclusions.

c) The random sample is a true representation of the entire population.

d) Any errors in the random sample are random.

There are errors in any measurements made by a human or made by an instrument a human manufactured. Until man reaches perfection, this will be true. If a surveyor believes his measurements are without error, he is wrong. By analyzing the measurements and acknowledging the presence of error, the surveyor will become more confident in his ability to exercise his professional judgment.

DEFINITIONS

Count: An exact quantity of a sample or population.

Measurement: An estimate of a quantity.

Error: The difference between an observed or computed value and the true value.

Systematic Error: an error that behaves according to a known theory but may vary in magnitude.

Random Error: An error whose presence is unavoidable and unpredictable but generally behaves according to mathematical laws and tend to cancel.

Precision: The agreement of the quality of a sample with itself.

Accuracy: The agreement of the quality of a sample with the true value.

Direct Measurement: A measurement made directly between two or more points.

Indirect Measurement: A computed measurement between points.

Significant Figures: The meaningful exactness of a number.

Sample Size: The number of observations or measurements in a sample (represented by the letter n in this paper).

Mean: Or arithmetic mean, the sum of the observations or measurements of a sample divided by the sample size

$$\left(\frac{\sum x_i}{n}\right)$$

or

$$\bar{x}$$

(note that x_i represents any value of the sample).

Median: The middle value of the sample.

Mode: The value which occurs most frequently in a sample.

Residual: The arithmetic difference between an individual value in a sample and the mean of that sample .

$$v_i = x_1 - \bar{x}$$

Standard Deviation: The number that, given the mean value, any observation in a sample will be in the range of the mean plus the standard deviation and the mean minus the standard deviation for a given confidence level. The confidence level is a mathematically predictable "goodness" of this range and represents the probability that any single value of a given sample chosen at random will fall within the previously mentioned range.

(Standard Deviation =

$$\sigma = \sqrt{\frac{\sum (v_i^2)}{n-1}}$$

(61)

Confidence Levels: Based upon mathematical theory, which is beyond the scope of this paper, that given any single value chosen at random from a sample, the confidence level is the probability of that single value will fall within the range specified previously. Approximate confidence levels are as follows:

σ---68.3%

(0.6745) σ---50%

(2) σ---95%

(2.5) σ---99%

(3) σ---99.7%

A field crew is asked to measure a distance with an electronic Distance Meter Instrument. Ignoring instrument centering error (How well the instrument/tribrach may be plumbed over a point), target centering error (similar to the instrument centering error), slope corrections, atmosphere corrections and the remote possibility (!) that the EDMi could be out of calibration, let us review their field data and analyze it.

The following 48 distances were measured:

950.275	950.241	950.293	950.264
950.243	950.250	950.232	950.262
950.263	950.232	950.320	950.289
950.251	950.242	950.274	950.313
950.252	950.261	950.250	950.241
950.231	950.252	950.249	950.265
950.289	950.220	950.181	950.254
950.310	950.200	950.201	950.243
950.250	950.242	950.239	950.250
950.264	950.263	950.274	950.221
950.219	950.275	950.249	950.231
950.243	950.264	950.250	950.252

Mean Distance

$$\frac{\sum x_i}{n} = \overline{x_i} = 950.253\ (950.252687500)$$

Standard Deviation

$$\sigma = \sqrt{\frac{\sum (v_i^2)}{n-1}} = 0.027\ (0.027348932)$$

Let us now further analyze this data and graphically illustrate what we have found.

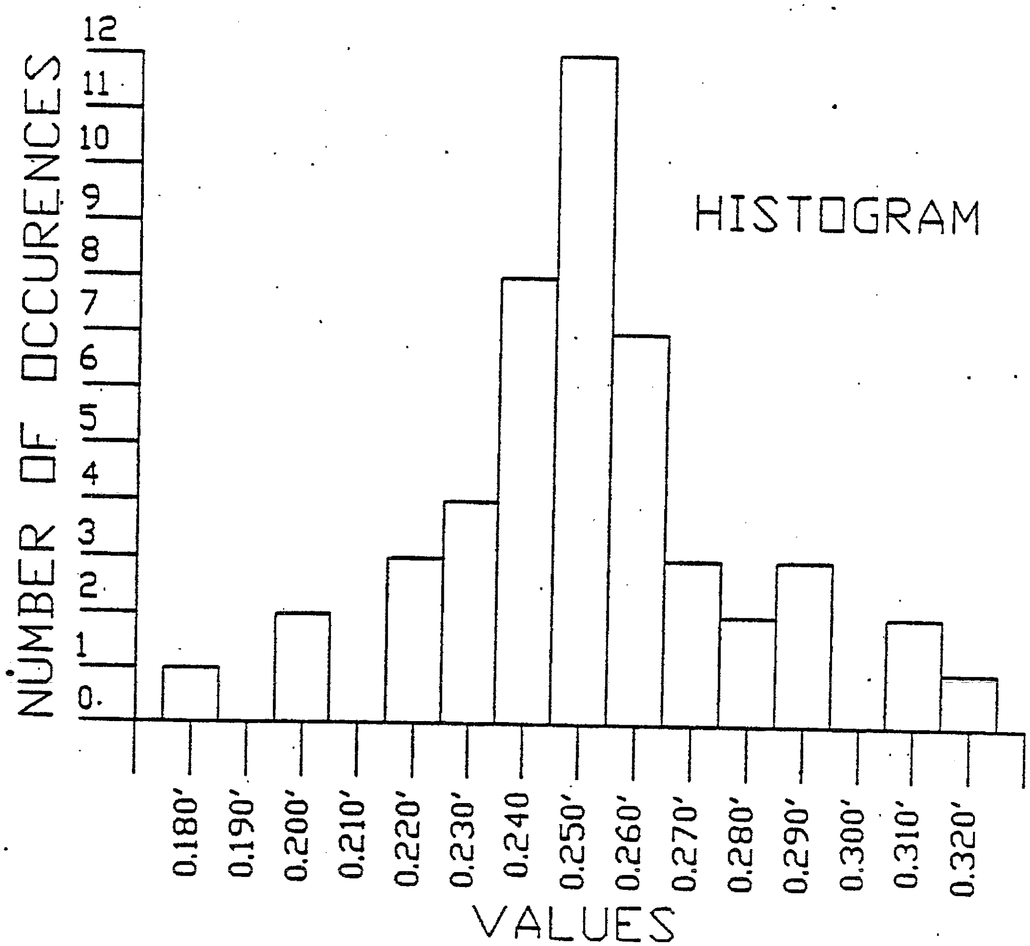
The smallest observation was 950.181' and the largest observation was 950.320'. This represents the range of the sample.

Now, group the data into columns with a range of 0.010', dropping the 950 for simplicity.

175	185	195	205	215	225	235	245	255	265	275	285	295	305	315
184	194	204	214	224	234	244	254	264	274	284	294	304	314	324
181		200		219	231	243	251	263	274	275	289		310	320
		201		220	232	243	252	264	274	275	293		313	
				221	232	241	250	261	265		289			
					231	242	250	263						
						242	252	264						
						239	250	264						
						241	249	262						
						243	249							
							250							
							254							
							252							
							250							
1	0	2	0	3	4	8	12	7	3	2	3	0	2	1

The above row of numbers being the sum of the number of observations in each column.

Now, make a HISTOGRAM (similar to a bar graph) of the data, using the grouping to $\pm 0.010'$ in the x direction and the frequency of occurrence in the y direction.

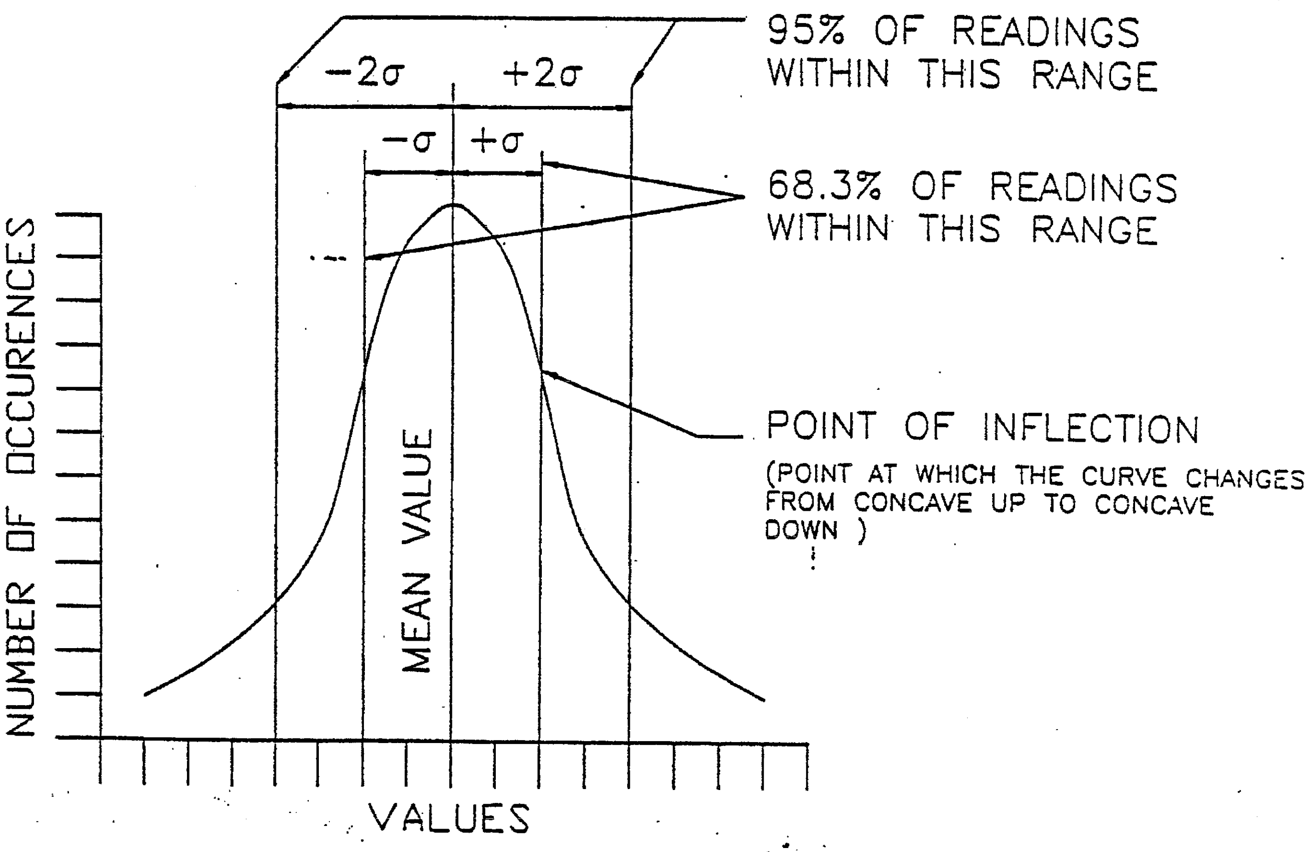


Note the following:

	± 0.01	± 0.001
Median	950.25'	950.250'
Mode	950.25'	950.250'

Although these values are the same in this sample, note they could be different in another.

A Probability Curve is a curve that is symmetric about the mean, has inflection points at $\pm\ \sigma$ and approaches the x-axis at $\pm$ infinity (∞). It is also called a Bell or a Gauss curve. The area underneath the curve represents probability and the total area is 100%. The area from $\pm\ \sigma$ divided by the total area is 68.3% of the total area. The area from $\pm\ 2\ \sigma$ divided by the total area is 95%. Note, the relationship to the level of certainty. The standard deviation, or one σ, is the confidence one has that, if the random sample fits the criteria outlined on page 1, any distance measured (in the vicinity of 950 feet) will be within $\pm\ \sigma$ of the true distance (assuming that the true = mean) 68.3% of the time.



The previous page discusses any value in the random sample and its level of certainty of being within $\pm\sigma$ of the mean. But the question may arise of how good the sample is itself? Are there "wild" values? There is a test that will evaluate the level of certainty of the mean with respect to the true value. It is the <u>Standard Error of the Mean</u>. This value carries with it the same level of certainty as the sigma (σ) value used to compute it.

The standard error of the mean, for one sigma, is

$$\sigma_{\bar{x}} = \pm \frac{\sigma}{\sqrt{n}} \ (68.3\% \textit{Probability})$$

For two sigma

$$2\sigma_{\bar{x}} = \pm \frac{2\sigma}{\sqrt{n}} \ (95\% \textit{Probability})$$

What this says is that the mean has a 68.3% probability (or whichever the level of certainty you choose in the computation) of being within $\pm\ \sigma_{\bar{x}}$ of the unknown true value.

For the previous values computed.)

$$\sigma_{\bar{x}} = \pm \frac{0.027}{\sqrt{48}} = \pm 0.004$$

$$2\sigma_{\bar{x}} = \pm \frac{0.0547}{\sqrt{48}} = \pm 0.008$$

For the set of Distances used in the example on Page 4, we may say, with 95% confidence, that the mean value, 950.253', is within ± 0.008' of the unknown true value. One could go on to say that to be within 0.008' of the true distance, given the range of distances this particular EDMi measured, one would have to measure this distance 48 times. This leads to another question. Given the EDMi's tested confidence level (at 95%) to
measure a group of distances and to be 95% certain that 95% of them were within 0.054' of the mean (2 σ), how many distances must one measure to be 95% confident that the mean is within 0.01' of the true distance?

$$2\sigma_{\bar{x}}=0.010'=\frac{0.054}{\sqrt{n}}$$

$$n=\left(\frac{0.054}{0.01}\right)^2=29.91$$

OR 30 TIMES

This is how specifications are written. First, the instrument is tested for standard deviation. One can then assume that this instrument will always carry this statistic as long as it is maintained in the same manner it was tested in and if the same user (human) is operating it (in the case of a theodolite). Then a parameter is given to you, say all the distances will be measured within 0.01' of the true distance or in the case of a theodolite, all angles will be turned within 5" of their true value. What the parameter is actually saying is the specification of standard error of the mean. One then takes their previously tested instrument (and perhaps user) with a standard deviation and determines the number of measurements (or repetitions of angles) necessary to achieve the parameter or accuracy specification!

For taping measurements, let us say that a particular crew (head & rear chainmen) was tested on a baseline course of 100 feet to have a 95% standard deviation of measurement that distance to within 0.01' (2 σ = 0.02'). They were then asked to measure 500.00' in 100 foot lengths. What is the standard (random) error they could expect (ignoring all other sources of error!)?

Error in a Series =

$$\sigma_y=\sigma_x\sqrt{n}$$

(where n is the number of times this error is introduced.)

$$\sigma_y=\pm0.02\sqrt{5}=0.045'$$

or they could expect 0.04' random error in that distance. This is an example of how errors <u>propagate.</u> They do not just sum. They tend to compensate.

The next question might be how many times must these two gentlepeople measure this course to be 95% confident that their result is within 0.01' of the true distance?

$$\sigma_{\bar{x}}=\pm0.01'=\frac{0.045}{\sqrt{n}}$$

$$n=\left(\frac{0.045}{0.01}\right)^2=20$$

OR 20 TIMES

To be within 0.02' (at 95% confidence level)

$$\sigma_{\bar{x}}=\pm0.02=\frac{0.045}{\sqrt{n}}$$

$$n=\left(\frac{0.045}{0.02}\right)^2=5$$

OR 5 TIMES

You can see to get "double" the accuracy does not mean to double the number of repetitions you do something.